Hybrid Perturbation/Bubnov-Galerkin Technique for Nonlinear Thermal Analysis

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A two-step hybrid analysis technique is presented for predicting the nonlinear steady-state temperature distribution in structures and solids. The technique is based on the successive application of the regular perturbation expansion and the classical Bubnov-Galerkin approximation. The functions associated with the various-order terms in the perturbation expansion of the temperature are first obtained by using the regular perturbation method. These functions are selected as coordinate functions (or temperature modes) and the classical Bubnov-Galerkin technique is then used to compute their amplitudes. The potential of the proposed hybrid technique for the solution of nonlinear thermal problems is discussed. The effectiveness of this technique is demonstrated by means of three numerical examples that include the effects of conduction, convection, and radiation modes of heat transfer. Results of the study indicate that the hybrid technique overcomes two major drawbacks of the classical techniques: 1) the requirement of using a small parameter in the regular perturbation method and 2) the arbitrariness in the choice of the coordinate functions in the Bubnov-Galerkin technique. Therefore, the proposed technique extends the range of applicability of the regular perturbation method and enhances the effectiveness of the Bubnov-Galerkin technique.

Nomenclature

A	= cross-sectional area of fin
$\mathcal{B}$	= boundary differential operator [see Eq. (2)]
c	= circumference of fin cross section
C, C_1, C_2, C_3	= constants defined in Appendix
e	= error norm, defined in Eq. (17)
h	= convective heat-transfer coefficient
k	= thermal conductivity coefficient
k_a, k_0	= thermal conductivity coefficients at $T=T_a$ and $T=0$
L	= fin length (also, side length of plate)
$\mathcal{L}, \bar{\mathcal{L}}, \tilde{\mathcal{L}}$	= differential operators defined in Eqs. (1), (3), and (4)
$\bar{\mathcal{L}}_0, \bar{\mathcal{L}}_1, \tilde{\mathcal{L}}_{00}$; $\tilde{\mathcal{L}}_1, \tilde{\mathcal{L}}_j', \tilde{\mathcal{L}}_{ij}'$	= differential operators defined in Eqs. (5), (6), and (9-12)
m	= number of points at which numerical values of temperature are evaluated
n	= total number of coordinate functions
n_1	= range of indices [see Eq. (8)]
q, q_1, q_2	= perturbation parameters
$\mathcal{R}_j, \mathcal{R}_{ij}$	= functions defined in Eqs. (10) and (12)
T	= temperature
$T_i, T_{i,j}$	= coordinate functions (temperature modes) defined in Eqs. (7) and (8)
T_m	= amplitude of sinusoidal temperature variation (see Fig. 1)
x_1, x_2	= Cartesian coordinates
β, λ	= parameters (see Table 1)
Γ	= boundary of the structure (or solid)
γ, γ_1	= nonlinear conductivity coefficients (see Table 1)
$\bar{\gamma}$	= constant (see Table 1)
ϵ	= emissivity of fin surface
θ	= nondimensional temperature (see Table 1)
θ_e	= first eigenfunction for zero-order perturbation equation
$\theta_i, \theta_{i,j}$	= dimensionless coordinate functions defined in Appendix
μ	= radiation-conduction parameter (see Table 1)
ξ, ξ_1, ξ_2	= dimensionless coordinates (see Table 1)
$\bar{\xi} \equiv 1-\xi$	
σ	= Stefan-Boltzmann constant
ψ_e	= amplitude of eigenfunction
$\psi_i, \psi_{i,j}$	= undetermined coefficients (amplitudes of coordinate functions)
Ω	= domain of the structure (or solid)

Subscripts

a	= ambient
b	= fin base
s	= effective sink

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Introduction

IN recent years, increasing attention has been devoted to the application of approximate analytical techniques to nonlinear heat-transfer problems. Analytical techniques have the major advantage over numerical discretization techniques of providing physical insight into the nature of the solution of the problem. Moreover, analytical techniques can be used in conjunction with a partitioning scheme for the thermal analysis of individual components of practical (complex) structures. Two of the commonly used approximate analytical techniques are the regular perturbation method and the Bubnov-Galerkin technique. A review of the many applications of these techniques to thermal problems is given in Refs. 1 and 2.

The regular perturbation method consists of the development of the solution in terms of unknown functions with preassigned coefficients. The unknown functions are obtained by solving a recursive set of differential equations which are, in general, simpler than the original governing differential equation of the problem. By contrast, in the Bubnov-Galerkin technique, the temperature is sought in the form of a series of a priori chosen coordinate functions (or temperature modes) with unknown coefficients.

Despite their usefulness in solving nonlinear thermal problems, the aforementioned two techniques have a number of drawbacks. Regular perturbation techniques have two major drawbacks. The first stems from the fact that as the number of terms in the perturbation series increases, the mathematical complexity of the differential equations builds up rapidly. Therefore, for practical applications, the perturbation series has to be restricted to a few terms. The second drawback is the requirement of restricting the perturbation parameter to small values in order to obtain solutions of acceptable accuracy. From a practical point of view, the main shortcoming of the Bubnov-Galerkin technique is the difficulty of selecting good coordinate functions (or modes).

The aforementioned drawbacks of the regular perturbation technique were recognized and a number of remedial actions proposed. These included the use of a small number of terms (e.g., two or three) in the perturbation expansion and either 1) generating "mimic functions" that give accurate numerical estimates of the solution over the entire physical domain (see Ref. 3), or 2) applying a nonlinear transformation (e.g., the Shanks transformation, see Ref. 4) to estimate the solution as the number of terms goes to infinity. However, as is shown in Ref. 5, the success of these methods cannot be guaranteed in general, and the remedial actions may fail to produce satisfactory results.

The successful experience obtained with the hybrid numerical procedure that combines the finite element method with the Galerkin technique in Ref. 6 raises the question as to whether the use of a hybrid technique combining both the standard regular perturbation method and the classical Bubnov-Galerkin technique might overcome the major drawbacks of the two techniques and provide a more effective approximate analysis procedure. The present study focuses on this question. Specifically, the objectives of this paper are: 1) to present a hybrid perturbation/Bubnov-Galerkin technique for nonlinear thermal analysis and 2) to demonstrate the effectiveness of the proposed technique by means of numerical examples. The technique is described as a formal procedure without any attempt to justify it rigorously. However, the numerical results presented herein are intended to give the analysts some insight into the potential of the proposed technique and to stimulate research and development of the mathematical foundations necessary to realize this potential.

To sharpen the focus of the study, discussion is limited to nonlinear steady-state thermal analysis with continuous temperature fields in the space domain. Nonlinear conduction, convection, and radiation modes of heat transfer are considered. The hybrid technique presented herein is also expected to be particularly useful for nonlinear transient thermal problems.

Basic Idea of the Hybrid Technique

Governing Equations

The steady-state thermal response of a structure or solid can be described by the following differential equation and boundary conditions:

$$\mathcal{L}(T)=0 \quad \text{in} \quad \Omega \tag{1}$$

and

$$\mathcal{B}(T)=0 \quad \text{on} \quad \Gamma \tag{2}$$

where T is the temperature, $\mathcal{L}$ and $\mathcal{B}$ the differential operators, Ω the domain of the structure or solid, and Γ its boundary. The differential operator $\mathcal{L}$ includes nonlinear conduction, convection, and radiation terms.

The application of the hybrid perturbation/Bubnov-Galerkin technique to the solution of Eqs. (1) and (2) can be conveniently divided into two distinct steps: 1) generation of coordinate functions (or temperature modes) using the standard regular perturbation method; and 2) computation of the amplitudes of the coordinate functions via the Bubnov-Galerkin technique. The procedure is described in detail subsequently.

Generation of Coordinate Functions

For the purpose of generating the required coordinate functions (or temperature modes), the governing differential equation (1) is embedded in a single- or multiple-parameter family of equations of the form

$$\bar{\mathcal{L}}(T,q)=0 \tag{3}$$

for the single-parameter case and

$$\tilde{\mathcal{L}}(T,q_1,q_2)=0 \tag{4}$$

for the two-parameter case, where q, q_1, and q_2 are normalizing or perturbation parameters. Extension to more than two parameters is straightforward and is not discussed herein.

For nonlinear thermal problems, it is convenient to choose the perturbation parameters such that the operators $\bar{\mathcal{L}}$ and $\tilde{\mathcal{L}}$ can be decomposed as follows:

$$\bar{\mathcal{L}}(T,q)=\bar{\mathcal{L}}_0(T)+\bar{\mathcal{L}}_1(T,q) \tag{5}$$

and

$$\tilde{\mathcal{L}}(T,q_1,q_2)=\tilde{\mathcal{L}}_{00}(T)+\tilde{\mathcal{L}}_1(T,q_1,q_2) \tag{6}$$

where $\bar{\mathcal{L}}_0$ and $\tilde{\mathcal{L}}_{00}$ are *linear* differential operators independent of q, q_1, and q_2; $\bar{\mathcal{L}}_1$ and $\tilde{\mathcal{L}}_1$ are *nonlinear* operators in T and its spatial derivatives.

The temperature T is represented by the regular perturbation expansion

$$T=\sum_{i=0}^{n-1} q^i T_i \tag{7}$$

for the single-parameter case, and

$$T=\sum_{j=0}^{n_1}\sum_{i=0}^{j} q_1^{j-i} q_2^i T_{i,j-i} \tag{8}$$

for the two-parameter case, where T_i and $T_{i,j}$ are perturbation functions that represent temperature modes and n is the total number of terms in the expansion. For the two-parameter case $n=\frac{1}{2}(n_1+1)(n_1+2)$. The differential equations used in generating T_i (or $T_{i,j}$) are obtained by substituting the temperature expansion, [Eq. (7) or (8)] into the governing differential equation (3) or (4) and setting the coefficients of q^i (or $q^i q^j$) successively equal to zero. This leads to the following recursive set of equations (for a detailed description of the regular perturbation technique, see any of the monographs on the subject, e.g., Refs. 7 and 8):

$$\bar{\mathcal{L}}_0(T_0)=0 \tag{9}$$

$$\bar{\mathcal{L}}_j'(T_j)=\bar{\mathcal{R}}_j \qquad j>0 \tag{10}$$

for the single-parameter case, and

$$\tilde{\mathcal{L}}_{00}(T_{0,0})=0 \tag{11}$$

$$\tilde{\mathcal{L}}_{ij}'(T_{i,j})=\tilde{\mathcal{R}}_{ij} \qquad (i\geq 0, j>0 \quad \text{or} \quad i>0, j\geq 0) \tag{12}$$

for the two-parameter case where $\bar{\mathcal{L}}_0$, $\tilde{\mathcal{L}}_{00}$, $\bar{\mathcal{L}}_j'$, and $\tilde{\mathcal{L}}_{ij}'$ are *linear* differential operators: $\bar{\mathcal{R}}_j$ depends on T_ℓ, with $\ell<j$; and $\tilde{\mathcal{R}}_{ij}$ depends on $T_{\ell,k}$ with $\ell\leq i$, $k<j$ or $\ell<i$, $k\leq j$.

Note that whereas the original differential equation (3) or (4), is nonlinear, the recursive set of Eqs. (9) and (10) or Eqs. (11) and (12) are *all linear*.

For prescribed nonzero values for the boundary temperatures or their spatial derivatives, the zero-order perturbation function T_0 (or $T_{0,0}$) is used to satisfy these nonhomogeneous boundary conditions. All of the higherorder perturbation functions, solutions of Eq. (10) or (12), satisfy only homogeneous boundary conditions.

Computation of Amplitudes of Coordinate Functions

The perturbation functions T_i (or $T_{i,j}$) are now chosen as coordinate functions and the temperature is expressed as a linear combination of these functions as follows:

$$T=\sum_{i=0}^{n-1}\psi_i T_i \tag{13}$$

for the single-parameter case, and

$$T=\sum_{j=0}^{n_1}\sum_{i=0}^{j}\psi_{i,j-i}T_{i,j-i} \tag{14}$$

for the two-parameter case where ψ_i and $\psi_{i,j}$ are unknown parameters representing the amplitudes of the coordinate functions (temperature modes) T_i and $T_{i,j}$, and n equals the total number of modes. In the two-parameter case $n = \frac{1}{2}(n_1+1)(n_1+2)$.

The parameters ψ_i (or $\psi_{i,j}$) are obtained by applying the Bubnov-Galerkin technique to the governing differential equation (3) or (4). The resulting set of n nonlinear equations can be cast in the following form:

$$\int_\Omega \bar{\mathcal{L}}\left(\sum_{i=0}^{n-1}\psi_i T_i\ ,\ q\right)T_j \mathrm{d}\Omega=0 \tag{15}$$

or

$$\int_\Omega \tilde{\mathcal{L}}\left(\sum_{j=0}^{n_1}\sum_{i=0}^{j}\psi_{i,j-i}T_{i,j-i}\ ,\ q_1\ ,\ q_2\right)T_{\ell,k}\mathrm{d}\Omega=0 \tag{16}$$

Note that for prescribed nonzero boundary conditions, the parameter ψ_0 (or $\psi_{0,0}$) is used to satisfy these boundary conditions and, therefore, the number of free parameters reduces to $n-1$.

Comments on the Selection of the Coordinate Functions

The chosen set of coordinate functions has the following two properties:

1) They are linearly independent and span the space of solutions in the neighborhood of the point of their generation. Therefore, they fully characterize the nonlinear solution in that neighborhood.

2) Their generation, using the regular perturbation technique, requires the solution of a recursive set of *linear* differential equations.

The first property is necessary for the convergence of the Bubnov-Galerkin approximation. The second property enhances the effectiveness of the proposed hybrid technique for solving nonlinear thermal problems.

The mathematical complexity of the governing differential equations used in generating the coordinate functions builds up rapidly with the increase in the number of these functions. Therefore, it is convenient to generate only a few coordinate functions and augment them, for example, by the first eigenfunction of the zero-order perturbation equation. The performance of the augmented set of coordinate functions will be discussed in the next section.

Numerical Studies

To evaluate the effectiveness of the proposed hybrid analysis technique, three nonlinear steady-state thermal problems were solved. For each problem the solutions obtained by the hybrid technique were compared with the regular perturbation expansions, other numerical approximations, and exact solutions, whenever available. Herein the results of three typical problems are discussed: 1) two-dimensional conduction in a square plate; 2) steady-state analysis of conducting-convecting fin with variable heat transfer coefficient; and 3) steady-state analysis of a one-dimensional conducting-convecting-radiating fin. In the first and third problems, the thermal conductivity is assumed to be temperature dependent. The three problems were analyzed by using the regular perturbation technique in Ref. 9. A summary of the differential equations and boundary conditions for the three problems is given in Table 1.

To assess the accuracy of the solutions obtained by the different techniques for each of the three problems, the temperature is evaluated at a large number of points in the solution domain. Then a vector of temperature errors $\{\Delta T\}$ is computed as follows:

$$\{\Delta T\}=\{T\}_e-\{T\}_{\text{approx}}$$

where $\{T\}_e$ and $\{T\}_{\text{approx}}$ are the vectors of *exact* and approximate temperatures, respectively.

A weighted Euclidean norm of $\{\Delta T\}$ is used as a global error measure, namely,

$$e=\frac{1}{m}\sqrt{\frac{\{\Delta T\}^t\{\Delta T\}}{\{T\}_e^t\{T\}_e}} \tag{17}$$

where superscript t denotes transposition and m is the number of points at which the numerical values of the temperature are evaluated. For the three problems analyzed, m was chosen to be 49, 30, and 30, respectively.

Two-Dimensional Steady Conduction in a Square Plate

The first problem considered is that of steady-state thermal conduction in a thin square plate with prescribed boundary temperatures (see Fig. 1). The thermal conductivity is assumed to vary linearly with temperature. An exact analytic solution based on the Kirchhoff transformation and a perturbation solution were presented in Ref. 9. As in Ref. 9, the perturbation parameter q was chosen in the present study to be the nonlinear conductivity coefficient γ (see Table 1).

Two sets of coordinate functions are considered in the hybrid technique. The first set uses the two terms of the perturbation expansion given in Ref. 9. In the second set, the two coordinate functions of the first set are augmented by the lowest eigenmode of the zero-order perturbation equation. The explicit forms of the coordinate functions are listed in the Appendix. Since one coordinate function (zero-order perturbation solution) is used in satisfying the nonzero prescribed boundary condition, the first set of coordinate functions allows the use of a single free parameter and the second set has two free parameters. The numerical values of the amplitudes of the coordinate functions in the hybrid technique are given in Table 2 for the two cases $q=1$ and 3. Note that the amplitudes of the first two coordinate functions in the regular perturbation technique are equal to 1 and q, respectively.

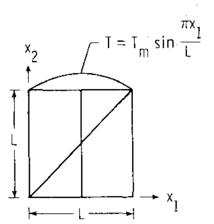
An indication of the accuracy of the solutions obtained using the hybrid technique and the perturbation method is given in Fig. 1. Also, Fig. 2 shows the error norm for four different values of the nonlinear conductivity coefficient (perturbation parameter), namely, $q=0.5$, 1.0, 3.0, and 5.0. An examination of the results shown in Figs. 1 and 2 reveals:

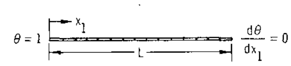
1) As expected, the accuracy of the perturbation solutions deteriorates rapidly with the increase in the perturbation parameter q (see Fig. 2), particularly for $q>1.0$. For $q=3.0$, the perturbation solution was considerably in error (see Fig. 1) and, for $q=5.0$, the error norm for the perturbation expansion could not be shown in Fig. 2.

2) The solutions obtained by using the hybrid technique with two coordinate functions (one free parameter) were

Table 1 Differential operators and boundary conditions used in numerical studies

	Square plate with temperature-dependent thermal conductivity	Conducting-convecting fin with variable heat-transfer coefficient	Conducting-covecting-radiating fin with variable thermal conductivity
$\mathfrak{L}(\theta,q)$ or $\mathfrak{L}(\theta,q_1,q_2)$	$(1+\gamma\theta)\left(\frac{\partial^2\theta}{\partial\xi_1^2}+\frac{\partial^2\theta}{\partial\xi_2^2}\right)+\gamma\left[\left(\frac{\partial\theta}{\partial\xi_1}\right)^2+\left(\frac{\partial\theta}{\partial\xi_2}\right)^2\right]$	$\frac{d^2\theta}{d\xi^2}-\lambda^2\theta^{1+\beta}$	$[1+\gamma_1(\theta-\theta_a)]\frac{d^2\theta}{d\xi^2}+\gamma_1\left(\frac{d\theta}{d\xi}\right)^2-\lambda^2(\theta-\theta_a)-\mu(\theta^4-\theta_s^4)$
Boundary conditions	$\theta(0,\xi_2)=\theta(\pi,\xi_2)=0$ $\theta(\xi_1,0)=0$ $\theta(\xi_1,\pi)=\sin\xi_1$	At $\xi=0,\quad \theta=1$ At $\xi=1,\quad \frac{d\theta}{d\xi}=0$	At $\xi=0,\quad \theta=1$ At $\xi=1,\quad \frac{d\theta}{d\xi}=0$
Perturbation parameter(s)	$q=\gamma$	Case 1: $q=\beta$ Case 2: $q=\lambda^2$	$q_1=\gamma_1$ $q_2=\mu$
Conductivity coefficient k	$k_0(1+\gamma\theta)$	k	$k_a[1+\gamma_1(\theta-\theta_a)]$
Convection coefficient h	—	$h_b\theta^\beta$	h
Definition of symbols	$\xi_\alpha=\frac{\pi x_\alpha}{L},\quad \alpha=1,2$ $\theta=\frac{T}{T_m}$	$\xi=\frac{x_1}{L}$ $\theta=\frac{T-T_a}{T_b-T_a},\quad T_a=0$ $\lambda^2=\frac{h_b cL^2}{kA}$ $h_b=\bar{\gamma}(T_b-T_a)^\beta$	$\xi=\frac{x_1}{L}$ $\theta=\frac{T}{T_b},\ \theta_a=\frac{T_a}{T_b}=0,$ $\theta_s=\frac{T_s}{T_b}=0$ $\lambda^2=\frac{hcL^2}{k_aA}$ $\mu=\frac{\sigma\epsilon T_b^3cL^2}{k_aA}$





reasonably accurate for $q\leq3.0$. For $q>3.0$ the hybrid technique predicts qualitatively the correct temperature distribution.

3) The accuracy of the solutions obtained by the hybrid technique was considerably improved when the two coordinate functions were augmented by the eigenmode. This is particularly true for $q\geq3.0$ (see Figs. 1 and 2).

Steady-State Analysis of a One-Dimensional Conducting-Convecting Fin with Variable Heat-Transfer Coefficient

The second problem is a straight conducting-convecting fin of length L, cross-sectional area A, and perimeter c exposed on both sides to a free convective environment of temperature T_a (see Fig. 3). The boundary conditions are a constant base temperature and an adiabatic tip. The thermal conductivity k is assumed to be independent of the temperature. The convective heat-transfer coefficient is taken to be of the form:

$$h=h_b\theta^\beta \tag{18}$$

where $\theta=(T-T_a)/(T_b-T_a)$ is a normalized temperature defined in terms of the ambient and fin-base temperatures T_a and T_b; β is a small parameter ($\beta=0.25$ and 0.33 for laminar

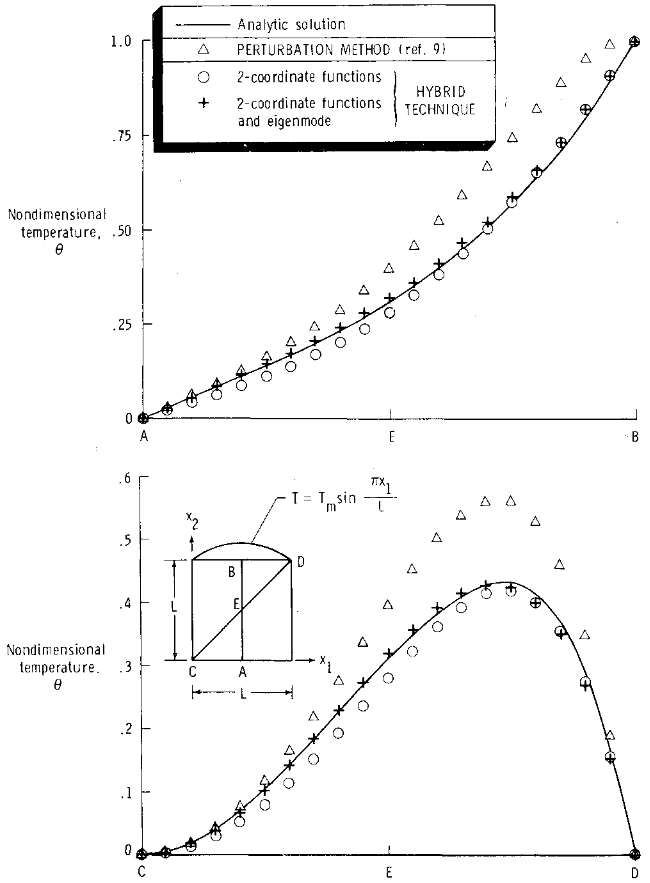


Fig. 1 Comparison of solutions obtained by perturbation method and hybrid technique for square plate with temperature-dependent thermal conductivity ($q=3.0$, $\theta=T/T_m$).

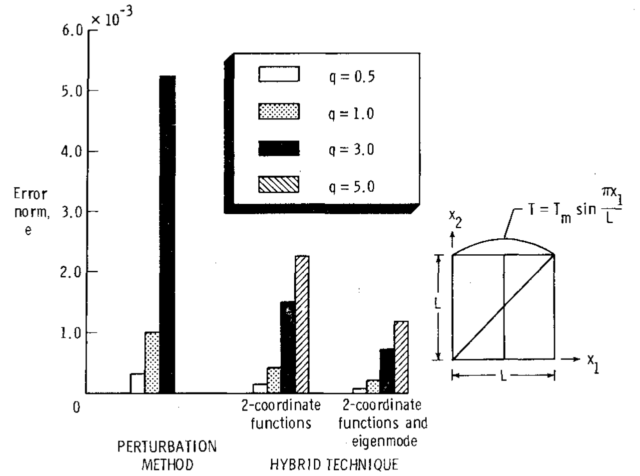


Fig. 2 Accuracy of solutions obtained by perturbation method and hybrid technique for square plate with temperature-dependent thermal conductivity (problem 1).

Table 2 Amplitudes of coordinate functions in Hybrid technique (square plate with temperature-dependent thermal conductivity, $q=\gamma$)

Amplitude	$q=1$		$q=3$	
	$n=2$	$n=3$	$n=2$	$n=3$
ψ_0	1.0	1.0	1.0	1.0
ψ_1	0.6856	0.6033	1.2278	0.9283
ψ_e[a]	—	0.0156	—	0.0581

[a] Amplitude of eigenfunction θ_e.

and turbulent conditions, respectively, see Ref. 9); and $h_b=\bar{\gamma}(T_b-T_a)^\beta$, where $\bar{\gamma}$ is a constant.

Two different choices were made for the perturbation parameter q. The first choice is the same as that of Ref. 9, namely $q=\beta$. The second choice is $q=\lambda^2=(h_b cL^2)/kA$, where λ^2 is a convection-conduction fin parameter (see Table 1). The latter choice resulted in significantly simplified expressions of the coordinate functions. For the case $q=\beta$, two coordinate functions were used and, for $q=\lambda^2$, four coordinate functions were generated. This corresponds to a single free parameter in the first case and three free parameters in the second. The expressions of the coordinate functions for the two cases are given in the Appendix.

An indication of the accuracy of the solutions obtained by the hybrid technique and the regular perturbation method are given in Fig. 3 for $\lambda=1.0$ and 2.0 and $\beta=0.33$ and 1.0. The latter value of β has no physical significance and was selected in order to amplify the effect of the magnitude of the perturbation parameter on the quality of the solution. The standard for comparison was taken to be the finite element solution obtained by using a uniform grid of 15 three-noded finite elements with quadratic Lagrangian interpolation functions for the temperature. As can be seen from Fig. 3, the accuracy of the perturbation solution is very sensitive to both the choice and the magnitude of the perturbation parameter. For $q=\beta$, the two-term perturbation expansion is accurate for $\beta\leq 0.33$, but becomes quite inaccurate for $\beta=1.0$. On the other hand, for $q=\lambda^2$, the four-term perturbation expansion is grossly in error for all $\lambda\geq 1.0$. The perturbation solutions for $\lambda=2$ could not be shown in Fig. 3. By contrast, the accuracy of the solutions obtained by the hybrid technique were found to be insensitive to the choice of the perturbation parameter. The solutions obtained by using $q=\beta$ and $q=\lambda^2$ were equally accurate and were almost indistinguishable from the finite element solutions.

Steady-State Analysis of a One-Dimensional Conducting-Convecting-Radiating Fin

The last problem considered is that of one-dimensional conduction in a straight fin of length L, cross-sectional area A, and perimeter c. Heat transfer from the surface of the fin involves both convection and radiation. The thermal conductivity is assumed to vary linearly with temperature and the convective heat-transfer coefficient and the surface emissivity are held constant. The boundary conditions consisted of a constant base temperature and an adiabatic tip.

The solution to this problem using a two-parameter regular perturbation technique was presented in Ref. 9. The two perturbation parameters were chosen to be the nonlinear conduction coefficient γ_1 and the radiation-conduction parameter μ (see Table 1). The six terms of the perturbation expansion of Ref. 9 were used as coordinate functions and, since one coordinate function was used in satisfying the prescribed nonzero boundary condition, only five parameters are left as free parameters. The expressions of the coordinate functions are given in the Appendix.

The accuracy of the solutions obtained with the hybrid technique and the two-parameter perturbation method is shown in Fig. 4 for three different values of the nonlinear conduction coefficient and the radiation-conduction parameter (perturbation parameters q_1 and q_2, respectively, $q_1=q_2=0.5$, 1.0, and 3.0). The standard of comparison is taken to be the finite element solution using a uniform grid of 15 three-noded finite elements with quadratic interpolation functions for the temperature. Also, the convergence of the solutions obtained using the hybrid technique with the increase in the number of coordinate functions is contrasted with the convergence of the perturbation technique in Fig. 5. As can be seen from Fig. 4, the solutions obtained using the perturbation method for $q_1=q_2=3$ are grossly in error. The error norms for the six-term perturbation expansion with

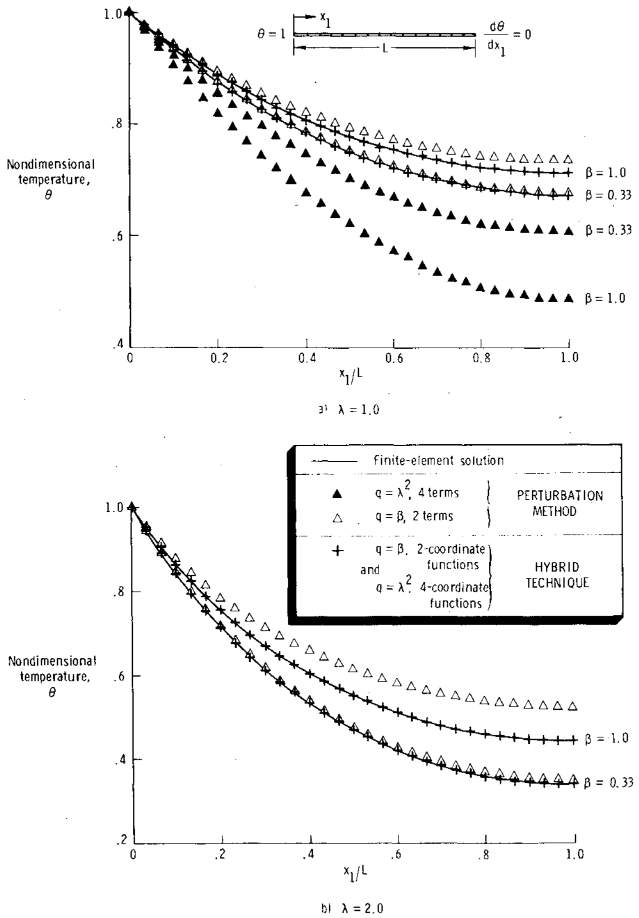


Fig. 3 Comparison of solutions obtained by perturbation method and hybrid technique for one-dimensional conducting-convecting fin with variable heat-transfer coefficient.

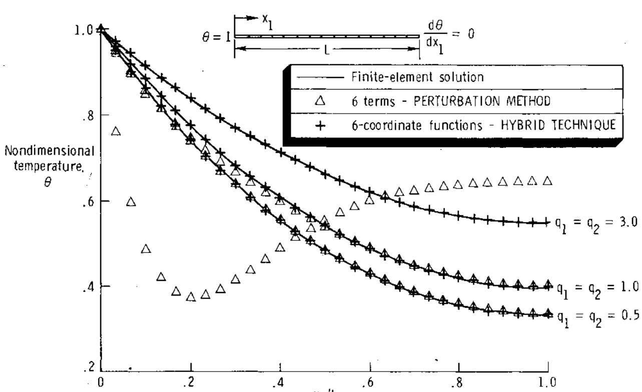


Fig. 4 Comparison of solutions obtained by perturbation method and hybrid technique for one-dimensional conducting-convecting-radiating fin with variable thermal conductivity ($\lambda = 2.0$, q_1 is the nonlinear conductivity coefficient and q_2 a radiation-conduction fin parameter.

$q_1 = q_2 = 3$ and 5 could not be shown in Fig. 5. By contrast, the solutions obtained using the hybrid technique are in close agreement with the finite element solution except for the small temperatures near the tip and, as can be seen from Fig. 6, the addition of the eigenmode to the coordinate functions resulted in reducing the errors near the tip. For $\lambda = 5.0$ and $q_1 = q_2 = 3.0$, the maximum error in the small temperature at the tip obtained by using six coordinate functions was 18.9%. This error reduced to 10.6% when the eigenmode was added.

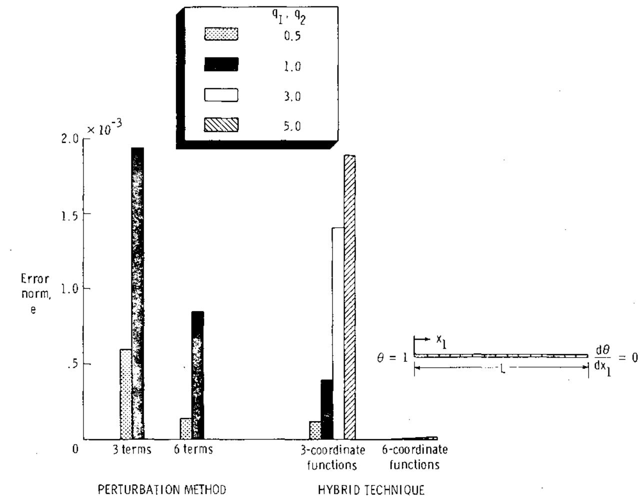


Fig. 5 Accuracy and convergence of solutions obtained by perturbation method and hybrid technique for one-dimensional conducting-convecting-radiating fin with variable thermal conductivity.

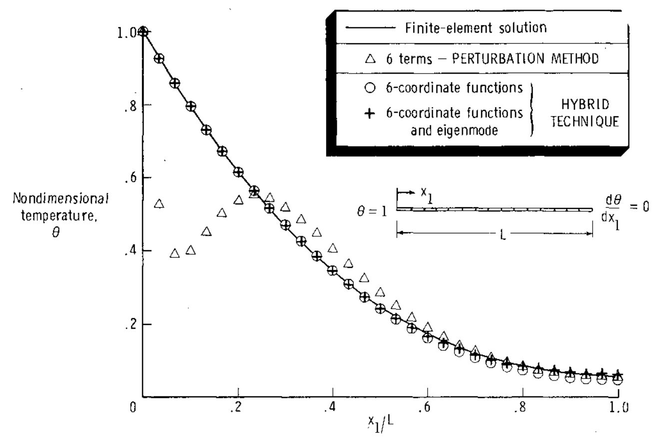


Fig. 6 Effect of adding eigenmode to coordinate functions on accuracy of solutions obtained by hybrid technique for conducting-convecting-radiating fin with variable thermal conductivity ($\lambda = 5.0$, $q_1 = q_2 = 3.0$).

Potential of the Proposed Hybrid Technique

The proposed hybrid analysis technique appears to have high potential for solution of nonlinear steady-state thermal problems. The numerical studies conducted clearly demonstrated the accuracy and effectiveness of the technique. In particular, the following two points are worth mentioning:

1) The proposed hybrid technique can be thought of as either of the following:

a) A generalized perturbation method in which 1) the perturbation expansion of the temperature contains free parameters rather than fixed coefficients; and 2) the perturbation parameters need not be small. Since the accuracy of the solutions obtained with the hybrid technique is insensitive to the choice of perturbation parameters, they may be introduced artificially to simplify the form of the recursive set of differential equations used in evaluating the various-order perturbation solutions.

b) An extended Bubnov-Galerkin approach with the coordinate functions generated by using the standard regular perturbation technique, rather than chosen a priori.

2) The hybrid technique presented herein is the analytic counterpart of the reduction method presented in Ref. 6. In the reduction method, the initial discretization is done via finite elements, then the vector of fundamental unknowns is expressed as a linear combination of a small number of global

temperature modes or basis vectors and the Bubnov-Galerkin technique is used to compute the coefficients in the linear combination. The primary objective of using the reduction method is to reduce considerably the number of degrees of freedom in the initial discretization and, hence, reduce the computational effort involved in the solution of the problem. By contrast, the objectives of the foregoing hybrid technique are: a) to enhance the effectiveness of the Bubnov-Galerkin technique by removing the arbitrariness in the selection of the coordinate functions; and b) to extend the range of validity of the regular perturbation method by removing the restriction of a small-perturbation parameter.

3) The hybrid technique can be applied, in conjunction with other numerical discretization techniques, to the nonlinear thermal analysis of practical structures with complicated geometries. To accomplish this, a partitioning scheme can be used with the structure divided into substructures. The substructures with simple geometry are analyzed using the foregoing hybrid analytical technique; and the substructures with complicated geometry (e.g., near the boundaries or interfaces) are analyzed using numerical discretization procedures.

Conclusions

A hybrid analysis technique based on the combined use of regular perturbation expansion and the classical Bubnov-Galerkin approximation is presented for predicting the nonlinear steady-state temperature distributions in structures and solids. The application of the technique to the solution of nonlinear thermal problems can be conveniently divided into the following two stages: 1) generation of coordinate functions (or temperature modes) using the standard regular perturbation method, and 2) approximating the temperature by a linear combination of these modes. The classical Bubnov-Galerkin technique is then used to compute the coefficients of the linear combination (amplitudes of the temperature modes).

Three numerical examples demonstrate the effectiveness of the hybrid technique for the solution of nonlinear steady-state thermal problems. The three problems are: 1) two-dimensional steady-state conduction in a square plate; 2) steady-state analysis of a one-dimensional conducting-convecting fin with variable heat-transfer coefficients; and 3) steady-state analysis of a one-dimensional conducting-convecting-radiating fin. In the first and third problems, the thermal conductivity is assumed to vary linearly with temperature.

The results suggest several conclusions relative to the choice of the coordinate functions and to the effectiveness of using the proposed hybrid technique in nonlinear steady-state thermal problems. These conclusions are as follows:

1) The proposed hybrid technique exploits the best elements of the regular perturbation method and the Bubnov-Galerkin technique as follows:

a) The regular perturbation method is used as a systematic and general approach for generating coordinate functions.

b) The Bubnov-Galerkin technique is used as an efficient procedure for minimizing and distributing the error in the temperature approximation throughout the domain.

2) The proposed technique greatly alleviates the following major drawbacks of the two classical techniques:

a) The requirement of using a small parameter in the regular perturbation expansion.

b) The arbitrariness in the choice of the temperature modes (or coordinate functions) in the Bubnov-Galerkin technique.

Therefore, the hybrid technique extends the range of applicability of the perturbation method and enhances the effectiveness of the Bubnov-Galerkin technique.

3) For a given number of coordinate functions, the accuracy of the solutions obtained by the hybrid technique can be improved by the addition of the first eigenmode of the zero-order perturbation equation to the coordinate functions. This is particularly true when the number of coordinate functions is small.

4) The accuracy of the solutions obtained by the hybrid technique is insensitive to the choice of the perturbation parameter(s). Therefore, the parameter(s) may be introduced artificially to simplify the form of the recursive set of differential equations used in evaluating the various-order perturbation solutions (viz, the coordinate functions).

Appendix: Coordinate Functions Used in Numerical Studies

The expressions for the coordinate functions used in the three problems are given in this Appendix. These expressions were generated (and checked) using the computerized symbolic manipulation system MACSYMA.[10] The different symbols used in the expressions are defined in Table 1.

Two-Dimensional Steady Conduction in Square Plate

The two coordinate functions are given by:

$$\theta_0 = \sinh\xi_2 \sin\xi_1 / \sinh\pi$$

$$\theta_1 = \sum_{m=1,3,\ldots}^{19} 2[m(m^2-4)\pi\sinh^2\pi]^{-1}$$

$$\times [1-\cosh 2\pi)\sinh m\xi_2/\sinh m\pi - (1-\cosh 2\xi_2)]\sin m\xi_1$$

The first eigenfunction for the zero-order perturbation equation is given by:

$$\theta_e = \sin\xi_1 \sin\xi_2$$

Steady-State Analysis of One-Dimensional Conducting-Convecting Fin with Variable Heat-Transfer Coefficient

The coordinate functions for the two cases $q=\beta$ and $q=\lambda^2$ are given by the following equations:

Case 1: $q=\beta$

$$\theta_0 = \operatorname{sech}\lambda \cosh\lambda\bar{\xi}$$

$$\theta_1 = C_1\cosh\lambda\bar{\xi} + C_2\bar{\xi}\sinh\lambda\bar{\xi} - C_3\Big[84+42(\lambda\bar{\xi})^2 + \frac{41}{12}(\lambda\bar{\xi})^4 + \frac{37}{360}(\lambda\bar{\xi})^6 + \frac{1}{560}(\lambda\bar{\xi})^8\Big]$$

where

$$C_1 = \frac{1}{2}\operatorname{sech}^2\lambda\Big(84+42\lambda^2+\frac{41}{12}\lambda^4+\frac{37}{360}\lambda^6+\frac{1}{560}\lambda^8\Big) - \frac{1}{2}\lambda\tanh\lambda\operatorname{sech}\lambda\,\ell n\operatorname{sech}\lambda$$

$$C_2 = \tfrac{1}{2}\lambda\operatorname{sech}\lambda\,\ell n\operatorname{sech}\lambda$$

$$C_3 = \tfrac{1}{2}\operatorname{sech}\lambda$$

$$\bar{\xi} = 1-\xi$$

Case 2: $q=\lambda^2$

$$\theta_0 = 1$$

$$\theta_1 = \tfrac{1}{2}(-1+\xi^2)\tfrac{1}{2}(-2\xi+\xi^2)$$

$$\theta_2=\frac{1}{12}(1+\beta)(8\xi-4\xi^3+\xi^4)$$

$$\theta_3=\frac{1}{8}(1+\beta)\left[-\frac{16}{5}(2+3\beta)\xi+\frac{8}{3}(1+\beta)\xi^3+2\beta\xi^4-\frac{2}{5}(1+4\beta)\xi^5+\frac{1}{15}(1+4\beta)\xi^6\right]$$

Steady-State Analysis of a One-Dimensional Conducting-Convecting-Radiating Fin

The six coordinate functions in this case are given by

$$\theta_{0,0}=\mathrm{sech}\lambda\ \cosh\lambda\bar{\xi}$$

$$\theta_{0,1}=\tfrac{1}{3}\mathrm{sech}^2\lambda[\cosh 2\lambda\ \mathrm{sech}\lambda\ \cosh\lambda\bar{\xi}-\cosh 2\lambda\bar{\xi}]$$

$$\theta_{1,0}=\frac{3}{8\lambda^2}\mathrm{sech}^4\lambda\left[-1+\mathrm{sech}\lambda\ \cosh\lambda\bar{\xi}\left(1-\frac{4}{9}\cosh 2\lambda-\frac{1}{45}\cosh 4\lambda\right)+\frac{4}{9}\cosh 2\lambda\bar{\xi}+\frac{1}{45}\cosh 4\lambda\bar{\xi}\right]$$

$$\theta_{0,2}=\frac{1}{6}\mathrm{sech}^3\lambda\left[\cosh\lambda\bar{\xi}\left(\frac{4}{3}\mathrm{sech}^2\lambda\ \cosh^2 2\lambda-\frac{9}{8}\mathrm{sech}\lambda\ \cosh 3\lambda-\frac{1}{2}\lambda\tanh\lambda\right)-\cosh 2\lambda\bar{\xi}\left(\frac{4}{3}\mathrm{sech}\lambda\ \cosh 2\lambda\right)+\frac{9}{8}\cosh 3\lambda\bar{\xi}+\frac{1}{2}\lambda\bar{\xi}\ \sinh\lambda\bar{\xi}\right]$$

$$\theta_{1,1}=\frac{1}{2\lambda^2}\mathrm{sech}^5\lambda\left\{-\cosh 2\lambda\ \mathrm{sech}\lambda+\mathrm{sech}\lambda\ \cosh\lambda\bar{\xi}\times\left[\frac{1}{2}\mathrm{sech}\lambda\ \cosh 2\lambda\left(3-\frac{4}{3}\cosh 2\lambda-\frac{1}{15}\cosh 4\lambda\right)+\frac{103}{320}\cosh 3\lambda+\frac{13}{576}\cosh 5\lambda+\frac{9}{24}\lambda\sinh\lambda\right]-\frac{1}{2}\mathrm{sech}\lambda\ \cosh 2\lambda\bar{\xi}\left(1-\frac{4}{3}\cosh 2\lambda-\frac{1}{45}\cosh 4\lambda\right)-\frac{103}{320}\cosh 3\lambda\bar{\xi}+\frac{1}{45}\mathrm{sech}\lambda\ \cosh 2\lambda\ \cosh 4\lambda\bar{\xi}-\frac{13}{576}\cosh 5\lambda\bar{\xi}-\frac{9}{24}\lambda\bar{\xi}\ \sinh\lambda\bar{\xi}\right\}$$

$$\theta_{2,0}=\frac{3}{4\lambda^4}\mathrm{sech}^7\lambda\left\{\frac{1}{43200}\sinh\lambda\bar{\xi}\{\mathrm{sech}\lambda(\sinh 3\lambda\bar{\xi}\times[9000-4000\cosh 2\lambda-200\cosh 4\lambda]+\sinh 5\lambda\bar{\xi}[1080-480\cosh 2\lambda-24\cosh 4\lambda])-12960\sinh 2\lambda\bar{\xi}-120\sinh 4\lambda\bar{\xi}+480\sinh 6\lambda\bar{\xi}+15\sinh 8\lambda\bar{\xi}-22680\lambda\bar{\xi}\}-\frac{1}{8640}\cosh\lambda\bar{\xi}[216\cosh 2\lambda\bar{\xi}+240\cosh 4\lambda\bar{\xi}+88\cosh 6\lambda\bar{\xi}+3\cosh 8\lambda\bar{\xi}]+\mathrm{sech}\lambda\left[\left(\frac{1}{36}\sinh^2\lambda\bar{\xi}-\frac{2}{225}\cosh^6\lambda\bar{\xi}\right)(45-20\cosh 2\lambda-\cosh 4\lambda)\right]+\frac{1}{2}\cosh^5\lambda\bar{\xi}\right\}+C\cosh\lambda\bar{\xi}$$

where

$$+C=\frac{1}{57600\lambda^4}\mathrm{sech}^8\lambda\{\sinh\lambda[22680\lambda+1290\sinh 2\lambda+120\sinh 4\lambda-480\sinh 6\lambda-15\sinh 8\lambda-\mathrm{sech}\lambda\ \sinh 3\lambda(9000-4000\cosh 2\lambda-200\cosh 4\lambda)-\mathrm{sech}\lambda\ \sinh 5\lambda(1080-480\cosh 2\lambda-24\cosh 4\lambda)]+\cosh\lambda[1080\cosh 2\lambda+1200\cosh 4\lambda+440\cosh 6\lambda+15\cosh 8\lambda]+\mathrm{sech}\lambda[\sinh^2\lambda(-54000+24000\cosh 2\lambda+1200\cosh 4\lambda)+\cosh^6\lambda(17280-7680\cosh 2\lambda-384\cosh 4\lambda]-21600\cosh^5\lambda\}$$

and

$$\bar{\xi}=1-\xi$$

The first eigenfunction of the zero-order perturbation equation is

$$\theta_e=\cos(\pi\bar{\xi}/2)$$

Note that the expression for $\theta_{2,0}$ is different from that presented in Ref. 9.

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